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Duality in Multistage Games

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1. INTRODUCTION

In [1] we have stated a class of multistage games and obtained necessary conditions for optimal pure strategies. In the present paper, we will state a new class of games named "dual games." Some of duality theorems will be obtained in multistage games. Detailed formulation of multistage games and necessary conditions for optimal pure strategies will be found in [1].

2. STATEMENT OF THE PROBLEM

We will discuss a class of multistage games whose playing space is a bounded region $\mathscr{C}$ of Euclidean $n + m$-space $\mathscr{D}$. The game is of two-person zero-sum with perfect information. Let $x_k = (x_k^1, x_k^2, \ldots, x_k^n)$ and $y_k = (y_k^1, y_k^2, \ldots, y_k^m)$ be the state of player A and B, respectively. $k \in \mathscr{K} = \{0, 1, 2, \ldots, \mathrm{N}\}$. Let $\mathscr{C}_x$ and $\mathscr{C}_y$ be bounded region of Euclidean n- and m-space respectively for which $\mathscr{C}$ is the direct product of them:

$$\mathscr{C} = \mathscr{C}_x \times \mathscr{C}_y .$$

Let $\mathscr{E}_x$ and $\mathscr{E}_y$ be Euclidean r- and s-spaces, respectively. The game is governed by the recurrence equations

$$x_{k+1} = f_k(x_k, u_k), \qquad k \in \mathscr{K}' = \{0, 1, \ldots, \mathrm{N} - 1\} \tag{2.1}$$

$$y_{k+1} = g_k(y_k, z_k), \qquad k \in \mathscr{K}', \tag{2.2}$$

where x_0 and y_0 are given.

$$f_k : \mathscr{C}_x \times \mathscr{E}_x \to \mathscr{C}_x \qquad \text{and} \qquad g_k : \mathscr{C}_y \times \mathscr{E}_y \to \mathscr{C}_y$$

are $C^{(2)}$ mappings. $u_k \in \mathscr{E}_x$ and $z_k \in \mathscr{E}_y$ are strategic variables and vector-valued functions $u_k(x_k, y_k)$ and $z_k(x_k, y_k)$ are pure strategies of players at kth stage. Constraints on the choice of u_k are

$$R_k^i(x_k, u_k) \geq 0, \qquad i = 1, 2, \ldots, p, \qquad k \in \mathscr{K}' \tag{2.3}$$

and on the choice of z_k are

$$S_k^j(y_k, z_k) \geq 0, \qquad j = 1, 2, \ldots, q, \qquad k \in \mathscr{K}'. \tag{2.4}$$

We will assume that R_k and S_k are of class $C^{(2)}$ on $\mathscr{C}_x \times \mathscr{E}_x$ and $\mathscr{C}_y \times \mathscr{E}_y$, respectively. Define the payoff by

$$P(k; x_0, y_0, X, Y, U, Z) = G_N(x_N, y_N) + \sum_{k=0}^{N-1} F_k(x_k, y_k, u_k, z_k), \tag{2.5}$$

where G_N is the terminal payoff defined on the terminal surface $\mathscr{T}(x_N, y_N)$. We will denote this game by $\Gamma(k; x_k, y_k)$.

Let a pair of pure strategies $\{u_k^*(x_k, y_k)\}$ and $\{z_k^*(x_k, y_k)\}$ be playable [1] and $P(k; x_0, y_0, X, Y, U^*, Z^*)$ be the value of the payoff resulting from this pair. We will call these strategies optimal if for some function $V(X, Y, U^*, Z^*)$

$$P(k; x_0, y_0, X, Y, U, Z^*) \leqslant V(X, Y, U^*, Z^*) \leqslant P(k; x_0, y_0, X, Y, U^*, Z) \tag{2.6}$$

holds for all possible $(x_k, y_k) \in \mathscr{C}$. The value $V(X, Y, U^*, Z^*)$ is called the value of the game $\Gamma(k; x_k, y_k)$.

In order to avoid complication we will not distinguish notations of vectors from their transpose. Scalar product of vectors will not be denoted any symbols. Subscripts represent the stage number and superscripts are used for the components of each variable at each stage of the game. The positivity of a vector is defined as the positivity of each component.

Definition 1 (Primal Game). The game $\Gamma(k; x_k, y_k)$ which we have stated above is said to be the primal game. That is, player A wishes to maximize the payoff P by selecting the pure strategies $\{u_k(x_k, y_k)\}$, according to the recurrence equation (2.1) and constrained by (2.3). While player B wants to minimize the payoff P by choosing the pure strategies $\{z_k(x_k, y_k)\}$, according to (2.2) and contrained by (2.4).

Definition 2 (Dual Game). The game $\tilde{\Gamma}(k; x_k, y_k)$ that has the following properties is said to be dual with respect to the primal.

Player A wishes to minimize the payoff for the dual game

$$\begin{aligned}\tilde{P}(k; x_0, y_0, X, Y, U, Z, \lambda, \psi, \nu, \mu) \\ = P(k; x_0, y_0, X, Y, U, Z) + \sum_{k=0}^{N-1} \{\lambda_k(f_k - x_{k+1}) + \psi_k(g_k - y_{k+1}) \\ + \nu_k R_k + \mu_k S_k\},\end{aligned}$$

wher λ_k, ψ_k, ν_k, and μ_k are elements of E^n, E^m, E^p, and E^q, respectively. We will note that they are the Lagrangian multiplies in the primal game [1]. Constraints of the strategies $\{u_k(x_k, y_k)\}$ are the systems of equalities:

$$\lambda_{k-1} = \frac{\partial H_k}{\partial x_k} + \frac{\partial}{\partial x_k}(\nu_k R_k), \qquad k \in \mathscr{K}' \tag{2.7}$$

$$\lambda_{N-1} = \frac{\partial G_N}{\partial x_N} \tag{2.8}$$

$$\frac{\partial H_k}{\partial u_k} - \frac{\partial}{\partial u_k}(\nu_k R_k) = 0, \qquad k \in \mathscr{K}' \tag{2.9}$$

with $\nu_k \leqslant 0$. Here $H_k(F_k, \lambda_k, f_k, \psi_k, g_k) = F_k + \lambda_k f_k + \psi_k g_k$. While player B wishes to maximize the payoff $\tilde{P}$ by selecting the pure strategies $\{z_k(x_k, y_k)\}$ constrained by

$$\psi_{k-1} = \frac{\partial H_k}{\partial y_k} + \frac{\partial}{\partial y_k}(\mu_k S_k), \qquad k \in \mathscr{K}' \tag{2.10}$$

$$\psi_{N-1} = \frac{\partial G_N}{\partial y_N} \tag{2.11}$$

$$\frac{\partial H_k}{\partial z_k} + \frac{\partial}{\partial z_k}(\mu_k S_k) = 0, \qquad k \in \mathscr{K}' \tag{2.12}$$

with $\mu_k \leqslant 0$.

Note that (2.7) to (2.12) are the necessary conditions for optimal strategies for two players in the primal game [1]. Therefore, by the dual game we mean the game having the necessary conditions for the primal game, as its constraints.

We have stated a pair of games. In what follows we will state two pairs of extremization problems and then will interpret them as the game we have stated above. To simplify the discussion we will define the following vectors:

$$\begin{aligned}
&\xi_k = (x_k, u_k), && \xi = (X, U) \\
&\eta_k = (y_k, z_k), && \eta = (Y, Z) \\
&\lambda = (\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{N-1}), && \lambda_k = (\lambda_k^1, \lambda_k^2, \dots, \lambda_k^n), \\
&\psi = (\psi_0, \psi_1, \psi_2, \dots, \psi_{N-1}), && \psi_k = (\psi_k^1, \psi_k^2, \dots, \psi_k^m), \\
&\nu = (\nu_0, \nu_1, \nu_2, \dots, \nu_{N-1}), && \nu_k = (\nu_k^1, \nu_k^2, \dots, \nu_k^p), \\
&\mu = (\mu_0, \mu_1, \mu_2, \dots, \mu_{N-1}), && \mu_k = (\mu_k^1, \mu_k^2, \dots, \mu_k^q), \qquad k \in \mathscr{K}',
\end{aligned}$$

where

$$X = (x_0^1, x_0^2, \dots, x_0^n, x_1^1, x_1^2, \dots, x_1^n, \dots, x_N^1, x_N^2, \dots, x_N^n).$$

Other vectors have similar meanings. Consequently ξ is an $n(\mathrm{N}+1)+r\mathrm{N}$ dimensional composite vector. We will write the recurrence equations (2.1) and (2.2) in the following manner:

Let T_1 be a $C^{(2)}$ mapping of $(n(\mathrm{N}+1)+r\mathrm{N})$-vector ξ into an $n\mathrm{N}$-vector λ; that is $T_1(\xi)$ is an $n\mathrm{N}$-vector whose components are $C^{(2)}$ in ξ. More precisely

$$T_1(\xi) = (f_0 - x_1, f_1 - x_2, \dots, f_{N-1} - x_N) = 0.$$

Let T_2 be a $C^{(2)}$ mapping of η into an $m\mathrm{N}$-vector ψ, i.e.,

$$T_2(\eta) = (g_0 - y_1, g_1 - y_2, \dots, g_{N-1} - y_N) = 0.$$

Furthermore we will rewrite the constraints (2.3) and (2.4) as systems of the form

$$T_4(\xi) \leqslant 0 \qquad \text{and} \qquad T_3(\eta) \geqslant 0,$$

where

$$T_4(\xi) = (R_0, R_1, \dots, R_{N-1}) \qquad \text{and} \qquad T_3(\eta) = (S_0, S_1, \dots, S_{N-1}).$$

PRIMAL PROBLEM I. *Let $(\xi^*, \lambda^*, \nu^*)$ be fixed. Find an η^* that minimizes $P(\xi^*, \eta)$ constrained by*

$$T_2(\eta) = 0 \qquad \textit{and} \qquad T_3(\eta) \geqslant 0.$$

DUAL PROBLEM I. *Let $(\xi^*, \lambda^*, \nu^*)$ be fixed. Find an (η^*, ψ^*, μ^*) that maximizes the real-valued function*

$$\tilde{P}(\xi^*, \eta) = P(\xi^*, \eta) + \lambda^* T_1(\xi^*) + \psi T_2(\eta) + \nu^* T_4(\xi^*) + \mu T_3(\eta)$$

constrained by

$$P(\xi^*, \eta)_\eta + \lambda^* T_1(\xi^*)_\eta + \psi T_2(\eta)_\eta + \nu^* T_4(\xi^*)_\eta + \mu T_3(\eta)_\eta = 0$$

with $\mu \leqslant 0$. Here, $P(\xi^*, \eta)_\eta = (\partial P/\partial \eta)$.

PRIMAL PROBLEM II. *Let (η^*, ψ^*, μ^*) be fixed. Find a ξ^* that maximizes $P(\xi, \eta^*)$ subjects to the constraints*

$$T_1(\xi) = 0 \qquad \textit{and} \qquad T_4(\xi) \leqslant 0.$$

DUAL PROBLEM II. *Let (η^*, ψ^*, μ^*) be fixed. Find an $(\xi^*, \lambda^*, \nu^*)$ that minimizes the function $\tilde{P}(\xi, \eta^*, \lambda, \psi^*, \nu, \mu^*)$ constrained by*

$$P(\xi, \eta^*)_\xi + \lambda T_1(\xi)_\xi + \psi^* T_2(\eta^*)_\xi + \nu T_4(\xi)_\xi + \mu^* T_3(\eta^*)_\xi = 0.$$

3. Convexity and Concavity

In order to discuss the dual game, we must characterize some of the important properties of convex and concave functions.

Definition 3. Suppose $\mathscr{B}$ is a convex set of a finite dimensional vector space $\mathscr{E}$. Then a real-valued function $f(x)$ defined in $\mathscr{B}$ is said to be convex if for all x and x^* in $\mathscr{B}$,

$$f((1-\theta)x+\theta x^*) \leqslant (1-\theta)f(x)+\theta f(x^*)$$

holds for $0 \leqslant \theta \leqslant 1$. A real-valued function in $\mathscr{B}$ is concave if $-f(x)$ is convex.

Definition 4. A vector-valued function in $\mathscr{B}$ is said to be convex (concave) if each component is a convex (concave) function.

Lemma 1. *If $f(x)$ is convex and differentiable in $\mathscr{B}$, then*

$$f(x^*)-f(x) \geqslant f_x(x^*-x),$$

where $f_x = (\partial f/\partial x)$ evaluated at x, for all x and x^* in $\mathscr{B}$.

If $f(x)$ is concave and differentiable in $\mathscr{B}$, then

$$f(x^*)-f(x) \leqslant f_x(x^*-x)$$

for all x and x^* in $\mathscr{B}$.

Proof. We will prove only the first relation. From the definition of a convex function, for $0 \leqslant \theta \leqslant 1$,

$$f(x^*)-f(x) \geqslant \frac{f(x+\theta(x^*-x))-f(x)}{\theta}.$$

Since the relation remains true as $\theta \to 0$

$$f(x^*)-f(x) \geqslant \lim_{\theta\to 0}\frac{f(x+\theta(x^*-x))-f(x)}{\theta}$$
$$=f_x(x^*-x).$$

4. A Dual Game

In order to obtain some theorems on dual game we will pose some restrictions.

(H_1) Functions f_k and R_k are concave in x_k and u_k for all $k \in \mathscr{K}'$.

(H_2) Functions g_k and S_k are convex in y_k and z_k for all $k \in \mathscr{K}'$.

(H_3) Function P is concave in both of x_k and u_k for each fixed y_k and z_k, and convex in both of y_k and z_k for each fixed x_k and u_k, for all $k \in \mathscr{K}'$. Furthermore P is of class $C^{(2)}$.

(H_4) The constraint qualification is always satisfied.

LEMMA 2. *Let* (H_2), (H_3) *and* (H_4) *hold. If* η^* *minimizes* $P(\xi^*, \eta)$ *in the primal problem I, then there exist vectors* ψ^* *and* $\mu^* \leqq 0$, *such that* (η^*, ψ^*, μ^*) *is the solution of the dual problem I. Conversely, if* (η^*, ψ^*, μ^*) *maximizes* $\tilde{P}(\xi^*, \eta, \lambda^*, \psi, \nu^*, \mu)$ *in the dual problem I and if the matrix of second derivative* $\tilde{P}_{\eta\eta}$ *evaluated at* (η^*, ψ^*, μ^*) *has an inverse, then* η^* *minimizes* $P(\xi^*, \eta)$ *in the primal problem I. In both cases*

$$\min_{\eta} P(\xi^*, \eta) = \max_{\eta,\psi,\mu} \tilde{P}(\xi^*, \eta, \lambda^*, \psi, \nu^*, \mu).$$

PROOF.

$$\tilde{P}(\xi^*, \eta^*, \lambda^*, \psi^*, \nu^*, \mu^*) - \tilde{P}(\xi^*, \eta, \lambda^*, \psi, \nu^*, \mu) \tag{4.1}$$

$$= P(\xi^*, \eta^*) - P(\xi^*, \eta) + \lambda^* T_1(\xi^*) - \lambda^* T_1(\xi^*) + \psi^* T_2(\eta^*) - \psi T_2(\eta)$$
$$+ \nu^* T_4(\xi^*) - \nu^* T_4(\xi^*) + \mu^* T_3(\eta^*) - \mu T_3(\eta) \tag{4.2}$$

$$= P(\xi^*, \eta^*) - P(\xi^*, \eta) + \psi^* T_2(\eta^*) - \psi T_2(\eta) + \mu^* T_3(\eta^*) - \mu T_3(\eta) \tag{4.3}$$

$$= P(\xi^*, \eta^*) - P(\xi^*, \eta) - \psi T_2(\eta) - \mu T_3(\eta) \tag{4.4}$$

$$\geqslant P_\eta(\eta^* - \eta) - \psi T_2(\eta) - \mu T_3(\eta) \tag{4.5}$$

$$= P_\eta(\eta^* - \eta) + \psi T_2(\eta^*) - \psi T_2(\eta) + \mu T_3(\eta^*) - \mu T_3(\eta) - \psi T_2(\eta^*)$$
$$- \mu T_3(\eta^*) \tag{4.6}$$

$$\geqslant (P_\eta + \psi T_2(\eta)_\eta - \mu T_3(\eta)_\eta)(\eta^* - \eta) - \psi T_2(\eta^*) - \mu T_3(\eta^*) \tag{4.7}$$

$$= - \psi T_2(\eta^*) - \mu T_3(\eta^*) \tag{4.8}$$

$$= - \mu T_3(\eta^*) \tag{4.9}$$

$$\geqslant 0.$$

From (4.4) to (4.5), we have used the convexity property of P in η and from (4.6) to (4.7) we employed the convexity of the mapping T_2 and T_3 (see Lemma 1). From (4.7) to (4.8) we used the constraint (2.13). Note that

$$\lambda^* T_1(\xi^*)_\eta = 0 \qquad \text{and} \qquad \nu^* T_3(\xi^*)_\eta = 0.$$

Conversely, if (η^*, ψ^*, μ^*) solves the dual problem I, the Kuhn-Tucker optimality conditions must hold for the dual problem I [1, 2]:

Define the Lagrangian

$$\tilde{\Phi}_1(\xi^*, \eta, \lambda^*, \psi, \nu^*, \mu, \Pi) = \tilde{P}(\xi^*, \eta, \lambda^*, \psi, \nu^*, \mu) + \Pi(P_\eta + \lambda^* T_1(\xi^*)_\eta + \psi T_2(\eta)_\eta + \mu T_3(\eta)_\eta + \nu^* T_4(\xi^*)_\eta),$$

where Π is an $m(\mathrm{N}+1)+s\mathrm{N}$ dimensional vector. Henceforth the condition applied to η gives

$$\tilde{\Phi}^*_{1\eta} = 0,$$

which turns out to be

$$P^*_\eta + \lambda^* T^*_1(\xi^*)_\eta + \psi^* T^*_2(\eta)_\eta + \nu^* T^*_4(\xi^*)_\eta + \mu^* T^*_3(\eta)_\eta$$

$$= -\Pi^*(P^*_{\eta\eta} + \lambda^* T^*_1(\xi^*)_{\eta\eta} + \psi^* T^*_2(\eta)_{\eta\eta} + \nu^* T^*_4(\xi^*)_{\eta\eta} + \mu^* T^*_3(\eta)_{\eta\eta}), \tag{4.10}$$

where $\mu^* T^*_3(\eta)_\eta$ is the partial derivative of $\mu^* T_3(\eta)$ with respect to η evaluated at η^*. Note that $\mu^* T_3(\eta^*) = 0$ [1, 2]. Since the left-hand side of (4.10) is zero and since we have assumed that $\tilde{P}_{\eta\eta}$ has an inverse evaluated at (η^*, ψ^*, μ^*), we must conclude that

$$\Pi^* = 0. \tag{4.11}$$

The condition $\tilde{\Phi}^*_{1\mu}\mu^* = 0$ implies

$$\mu^* T_3(\eta^*) + \Pi^* \mu^* T^*_3(\eta)_\eta = 0.$$

Hence from (4.10)

$$\mu^* T_3(\eta^*) = 0.$$

With regard to the well-known theorem in nonlinear programming (see, for example, Theorem 3.5 of [3]), the other condition with respect to μ yields

$$T_3(\eta^*) + \Pi^* T^*_3(\eta)_\eta - \beta = 0$$

for some $m(\mathrm{N}+1)+s\mathrm{N}$ dimensional vector $\beta \geqslant 0$. But since $\Pi^* = 0$, we see that

$$T_3(\eta^*) \geqslant 0.$$

It is easy to see that $T_2(\eta^*) = 0$. This shows that η^* is in the feasible region of the primal problem I. Finally, $\mu^* \leqslant 0$ is the hypothesis (2.13). Thus all the conditions for the primal problem I hold. Q.E.D.

We have shown that

$$\min_\eta P(\xi^*, \eta) = \max_{\eta,\psi,\mu} \tilde{P}(\xi^*, \eta, \lambda^*, \psi, \nu^*, \mu)$$

and hence that

$$\tilde{P}(\xi^*, \eta^*, \lambda^*, \psi^*, \nu^*, \mu^*) = P(\xi^*, \eta^*) + \lambda^* T_1(\xi^*) + \psi^* T_2(\eta^*) + \nu^* T_4(\xi^*) + \mu^* T_3(\eta^*)$$
$$= P(\xi^*, \eta^*).$$

Lemma 3. *Let the assumptions* (H_1), (H_3) *and* (H_4) *hold. If* ξ^* *maximizes* $P(\xi, \eta^*)$ *in the primal problem II, there exist vectors* λ^* *and* $\nu^* \leqslant 0$, *such that* $(\xi^*, \lambda^*, \nu^*)$ *is the solution of the dual problem II. Conversely, if* $(\xi^*, \lambda^*, \nu^*)$ *minimizes* $\tilde{P}(\xi, \eta^*, \lambda, \psi^*, \nu, \mu^*)$ *in the dual problem II and if the matrix of second derivative* $\tilde{P}_{\xi\xi}$ *evaluated at* $(\xi^*, \lambda^*, \nu^*)$ *has an inverse, then* ξ^* *maximizes* $P(\xi, \eta^*)$ *in the primal problem II. Moreover in both cases*

$$\max_{\xi} P(\xi, \eta^*) = \min_{\xi,\lambda,\nu} \tilde{P}(\xi, \eta^*, \lambda, \psi^*, \nu, \mu^*).$$

Proof. From the concavity of the function P in ξ and that of T_1 and T_4,

$$\begin{aligned}
&\tilde{P}(\xi^*, \eta^*, \lambda^*, \psi^*, \nu^*, \mu^*) - \tilde{P}(\xi, \eta^*, \lambda, \psi^*, \nu, \mu^*)\\
&= P(\xi^*, \eta^*) - P(\xi, \eta^*) - \lambda T_1(\xi) - \nu T_4(\xi)\\
&\leqslant P_\xi(\xi^* - \xi) - \lambda T_1(\xi) - \nu T_4(\xi)\\
&\leqslant (P_\xi + \lambda T_1(\xi)_\xi + \nu T_4(\xi)_\xi)(\xi^* - \xi) - \lambda T_1(\xi^*) - \nu T_4(\xi^*)\\
&= - \nu T_4(\xi^*)\\
&\leqslant 0.
\end{aligned}$$

Thus

$$\tilde{P}(\xi^*, \eta^*, \lambda^*, \psi^*, \nu^*, \mu^*) = P(\xi^*, \eta^*)$$

and hence

$$\max_{\xi} P(\xi, \eta^*) = \min_{\xi,\lambda,\nu} \tilde{P}(\xi, \eta^*, \lambda, \psi^*, \nu, \mu^*).$$

Converse is easy to prove. Q.E.D.

Now from Lemma 2 and Lemma 3, we can show that in the primal game $\Gamma(k; x_k, y_k)$,

$$P(X, Y^*, U, Z^*) \leqslant V(X^*, Y^*, U^*, Z^*) \leqslant P(X^*, Y, U^*, Z)$$

holds and in the dual game $\tilde{\Gamma}(k; x_k, y_k)$

$$\tilde{P}(X^*, Y, U^*, Z, \lambda^*, \psi, \nu^*, \mu) \leqslant \tilde{V}(X^*, Y^*, U^*, Z^*, \lambda^*, \psi^*, \nu^*, \mu^*)$$
$$\leqslant \tilde{P}(X, Y^*, U, Z^*, \lambda, \psi^*, \nu, \mu^*)$$

holds. Thus we have proved the following two theorems:

THEOREM 1. *Let* (H_1) *to* (H_4) *hold. If playable strategies* $\{u_k^*(x_k, y_k)\}$ *and* $\{z_k^*(x_k, y_k)\}$ *are optimal pure strategies for the primal game* $\Gamma(k; x_k, y_k)$ *which has a unique value* $V(X^*, Y^*, U^*, Z^*)$, *then* $\{u_k^*(x_k, y_k)\}$ *and* $\{z_k^*(x_k, y_k)\}$ *are optimal pure strategies also for the dual game* $\tilde{\Gamma}(k; x_k, y_k)$, *having its value* $\tilde{V}(X^*, Y^*, U^*, Z^*, \lambda^*, \psi^*, \nu^*, \mu^*)$. *Furthermore*

$$V(X^*, Y^*, U^*, Z^*) = \tilde{V}(X^*, Y^*, U^*, Z^*, \lambda^*, \psi^*, \nu^*, \mu^*).$$

THEOREM 2. *Let all the assumptions of Theorem 1 hold. Let the matrices of second derivative* $\tilde{P}_{\xi\xi}$ *and* $\tilde{P}_{\eta\eta}$ *evaluated at* $(\xi^*, \lambda^*, \nu^*)$ *and* (η^*, ψ^*, μ^*) *respectively have their inverse. If playable strategies* $\{u_k^*(x_k, y_k)\}$ *and* $\{z_k^*(x_k, y_k)\}$ *are optimal for the dual game* $\tilde{\Gamma}(k; x_k, y_k)$ *which has a unique value* $\tilde{V}(X^*, Y^*, U^*, Z^*, \lambda^*, \psi^*, \nu^*, \mu^*)$, *then they are optimal pure strategies also for the primal game* $\Gamma(k; x_k, y_k)$ *having its value* $V(X^*, Y^*, U^*, Z^*)$. *Moreover*

$$\tilde{V}(X^*, Y^*, U^*, Z^*, \lambda^*, \psi^*, \nu^*, \mu^*) = V(X^*, Y^*, U^*, Z^*).$$

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